Visualizing the 3-Sphere

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Abstract

The 3-sphere S^3 is a fundamental object in geometry and topology, representing the set of points equidistant from the origin in four-dimensional space. Although it cannot be directly visualized in Euclidean 3-space, its structure can be understood through coordinate systems, projections, and analytic parametrizations derived from multivariable calculus. This paper introduces several techniques for visualizing S^3 and interpreting its curvature and topology through calculus and differential geometry.

1 Introduction

The 3-sphere S^3 is defined as

$$S^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}.$$

It serves as the natural generalization of the circle $S^1 \subset \mathbb{R}^2$ and the ordinary sphere $S^2 \subset \mathbb{R}^3$. Despite its four-dimensional embedding, calculus and geometry allow us to understand S^3 through analytic descriptions and lower-dimensional projections.

2 Parametrization via Angles

A convenient parameterization of S^3 uses three angular coordinates (θ, ϕ, ψ):

$$\begin{aligned} x &= \cos\theta\cos\phi, \\ y &= \cos\theta\sin\phi, \\ z &= \sin\theta\cos\psi, \\ w &= \sin\theta\sin\psi, \end{aligned} \quad \text{where } \theta \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right],\ \phi, \psi \in [0, 2\pi).$$

Each value of θ corresponds to a torus $S^1 \times S^1$ embedded in S^3. Thus S^3 can be visualized as a continuous family of linked tori shrinking to circles at the poles. This geometric decomposition is known as the *Hopf fibration.*

3 Stereographic Projection

To render S^3 in three dimensions, one uses stereographic projection from the north pole $(0, 0, 0, 1)$ onto the hyperplane $w = 0$:

$$\pi(x, y, z, w) = \left(\frac{x}{1-w}, \frac{y}{1-w}, \frac{z}{1-w}\right).$$

This mapping preserves smoothness and conformality, producing a 3D representation of S^3 in $\mathbb{R}^3$. Curves and tori under this projection appear as nested, linked shapes, visually expressing the Hopf structure.

4 Curvature and Differential Geometry

The induced metric on S^3 from $\mathbb{R}^4$ is

$$ds^2 = d\theta^2 + \cos^2\theta\, d\phi^2 + \sin^2\theta\, d\psi^2,$$

with constant sectional curvature $+1$. The Christoffel symbols and curvature tensors can be computed directly from this metric using multivariable calculus, confirming that S^3 is a Riemannian manifold of constant positive curvature.

5 Computational Visualization

In computer visualization, we approximate S^3 by sampling points in $\mathbb{R}^4$ satisfying $x^2 + y^2 + z^2 + w^2 = 1$, then projecting to $\mathbb{R}^3$ via stereographic projection. The mapping can be implemented as:

$$(x', y', z') = \frac{(x, y, z)}{1 - w}.$$

Color or transparency can encode the hidden w-dimension. This approach allows rendering linked tori and geodesics of S^3 in standard 3D visualization software.

6 Conclusion

The 3-sphere embodies a profound synthesis of algebra, calculus, and geometry. Through explicit parametrizations and projections, one can visualize its structure and curvature, revealing the hidden beauty of higher-dimensional manifolds in familiar 3D space. Calculus provides both the local analytic tools and the global invariants that make this visualization mathematically rigorous.

References

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